Distribution of Global Defensive k-Alliances over some Graph Products

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Abstract

If $G = (V_G, E_G)$ is a graph, then $S \subseteq V_G$ is a global defensive k-alliance in G if (i) each vertex not in S has a neighbor in S and (ii) each vertex of S has at least k more neighbors inside S than outside of it. The global defensive k-alliance number of G is the minimum cardinality among all global defensive k-alliance in G. In this paper this concept is studied on the generalized hierarchical, the lexicographic, the corona, and the edge corona product. For all of these products upper bounds expressed with related invariants of the factors are given. Sharpness of the bounds are also discussed.

Key words: global alliance; global defensive k-alliance, hierarchical product of graphs; lexicographic product of graphs

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1 Introduction

Alliances form a phenomena in many ways, say in politics, in relations between people (alliances such as common friendship), in social networks (say, Twitter users following each other), in natural sciences (say, animals from the same group), to list just a very small sample of examples. Since graphs are standard mathematical models for several of such phenomena, there is a strong need for a theory of alliances in graphs. The foundation for this theory was set up in [13], where alliances were classified into defensive, offensive, and powerful.

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In this paper we are interested in defensive alliances, the theory of which has been recently surveyed in [23]. More precisely, we are interested in global defensive k-alliances, a concept that widely generalizes global defensive alliances and goes back to the paper [16] published two years before [13]. We refer to [15, 20] for several mathematical properties of global defensive k-alliances and to [10, 13, 16] for applications of alliances in as different areas as national defence, studies of RNA structures, and fault-tolerant computing.

Since determining an optimal (global) defensive (k-)alliance is NP-hard, a way to approach the problem is via dynamic programming: decompose a given graph into smaller parts, solve the problem on these smaller graphs, and deduce a solution or an approximate the original problem from the obtained partial solutions. Graph products and similar operations are natural candidates for such an approach. In [6], the global defensive alliance was investigated on the join, the corona, and the composition (alias lexicographic product) of graphs. Very recently, the global defensive k-alliance was studied on the Cartesian product, the strong product, and the direct product of graphs [21, 22]. In this paper we continue this direction of research by investigating the global defensive k-alliance on additional graph products and graph operations, in particular extending some previous results.

In the rest of the introduction concepts and notation needed are introduced, in particular the global defensive k-alliance is formally defined. In the subsequent section we study the generalized hierarchical product, in Section 3 we proceed with the lexicographic product, while in Section 4 we consider the corona and the edge corona product.

Throughout this article, $G = (V_G, E_G)$ stands for a simple graph of order $n(G) = |V_G|$ and size $m(G) = |E_G|$. The degree of $v \in V_G$ is denoted by $\deg_G(v)$, and the minimum and the maximum degree of G by δ_G and Δ_G, respectively. If $X \subseteq V_G$, then the subgraph induced by X is denoted by $G\langle V' \rangle$. If $S \subseteq V_G$ and $v \in V_G$, then $N_S(v)$ is the set of neighbors of v in S, that is, $N_S(v) := \{u \in S \mid uv \in E_G\}$. The *complement* of S in V_G is denoted by $\bar{S}$.

Let $X \subseteq V_G$. Then $D \subseteq V_G$ *dominates* X, if every vertex from $X \setminus D$ has a neighbor in D. When $X = V_G$ we say that D is a *dominating set* of G. The *domination number* $\gamma(G)$ is the cardinality of a smallest dominating set of G. Let $G = (V_G, E_G)$ be a graph and $k \in \{-\delta_G, \ldots, \delta_G\}$. Then a nonempty set $S \subseteq V_G$ is a *global defensive k-alliance* in G if the following conditions hold:

1. S is a dominating set of G and

2. for every $v \in S$, $|N_S(v)| \geq |N_{\bar{S}}(v)| + k$.

When S fulfils Condition 2 it is a *defensive k-alliance*. To shorten the presentation, we will abbreviate global defensive k-alliance as GDk-A and defensive k-alliance as Dk-A. Note that GDk-A is a Dk-A which is also a dominating set.

The global defensive k-alliance number $\gamma_k^d(G)$ of G (abbreviated as *GDk-A number*) is the smallest order of a GDk-A in G. If G admits not a single GDk-A, we set $\gamma_k^d(G) = \infty$. Similarly, *The defensive k-alliance number $\gamma_k(G)$ of G* (abbreviated as *Dk-A number*) is the minimum cardinality among all Dk-As in G. If G does not contain a Dk-A set then we set $\gamma_k(G) = \infty$. Finally, we use the notation $[n] = \{1, \ldots, n\}$.

2 Generalized hierarchical products

A graph G together with a fixed vertex subset $U \subseteq V_G$ will be denoted by $G(U)$. If G and H are graphs, and $U \subseteq V_G$, then the *generalized hierarchical product* $G(U) \sqcap H$ is the graph with the vertex set $V_G \times V_H$, vertices (g, h) and (g', h') being adjacent if and only if either $g = g' \in U$ and $hh' \in E_H$, or $gg' \in E_G$ and $h = h'$. Note that when $U = V_G$, then the generalized hierarchical product is just the classical Cartesian product of graphs [8], that is, $G(V_G) \sqcap H = G \square H$. The generalized hierarchical product was introduced for the first time in [4], we also refer to [1, 2, 3, 17] for additional results on it as well as on its applications.

Consider $G(U) \sqcap H$ and let $h \in V(H)$. Then the set of vertices $\{(g, h) : g \in V_G\}$ is called a *G-layer over h*. Similarly an *H-layer over $g \in V_G$* is defined. Note that a G-layer over h induces a subgraph of $G(U) \sqcap H$ isomorphic to G and an H-layer over $g \in U$ induces a subgraph of $G(U) \sqcap H$ isomorphic to H.

Theorem 1. *If G and H are graphs and $U \subseteq V_G$, then*

$$\gamma_k^d(G(U) \sqcap H) \leq \gamma_k^d(G)n(H).$$

Proof. Let $U \subseteq V_G$, let S_G be a GDk-A in G with $|S_G| = \gamma_k^d(G)$, and set $S = S_G \times V_H$.

Note first that S is a domination set of $G(U) \sqcap H$. Indeed, since S_G is a dominating set of G, every G-layer over $h \in V_H$ is dominated by the intersection of S with the layer. So all G-layers are dominated by S and therefore $G(U) \sqcap H$ is dominated by S.

To show that S is a Dk-A in $G(U) \sqcap H$ consider an arbitrary vertex (g, h) from S. If $g \in U$, then we have

$$|N_S((g, h))| - |N_{\bar{S}}((g, h))| = (|N_{S_G}(g)| - |N_{\bar{S}_G}(g)|) + |N_{V_H}(h)|$$
$$\geq k + \deg_H(h)$$
$$\geq k,$$

and if $g \notin U$, then

$$|N_S((g, h))| - |N_{\bar{S}}((g, h))| = (|N_{S_G}(g)| - |N_{\bar{S}_G}(g)|) + 0 \geq k.$$

We have thus seen that S is a GDk-A. Since $|S| = \gamma_k^d(G)n(H)$, the argument is complete. $\qquad\square$

Let Γ be the graph of the truncated cube, see the right-hand side of Fig. 1. Then Γ can be represented as the hierarchical product $G(U) \sqcap P_2$, where G is the graph of order 12 on the left-hand side of Fig. 1 and $U = \{g_1, g_4, g_9, g_{12}\}$.
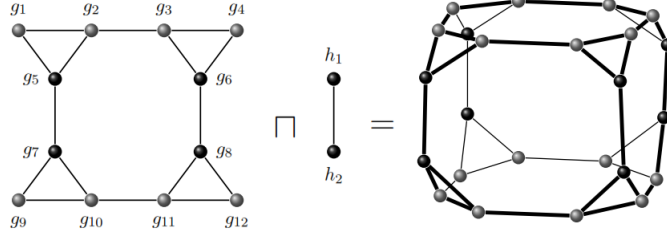

Figure 1: The graph $\Gamma = G(U) \sqcap P_2$, where $U = \{g_1, g_4, g_9, g_{12}\}$.

The set $S_G = \{g_5, g_6, g_7, g_8\}$ (drawn in black in Fig. 1) is a global defensive (-1)-alliance in G. Since $\gamma(G) = 4$ (which follows for instance from the fact that the vertices g_1, g_4, g_9, g_{12} are pairwise at distance at least 3), we have $\gamma_{-1}^d(G) = 4$. Hence Theorem 1 yields $\gamma_{-1}^d(G(U) \sqcap P_2) \le \gamma_{-1}^d(G)n(P_2) = 8$.

Now, let S_Γ be a minimal global defensive (-1)-alliance in Γ. Since Γ is a 3-regular graph and for every v of V_Γ we have $|N_{S_\Gamma}(g)| - |N_{\bar{S}_\Gamma}(g)| \ge -1$, each vertex u of S_Γ must have at least one neighbor in S_Γ. On the other hand, S_Γ dominates vertices of Γ and so $\gamma_{-1}^d(\Gamma) \ge 8$; hence the the inequality of Theorem 1 is sharp.

As already mentioned, $G(V_G) \sqcap H = G \square H$. Hence Theorem 1 for the case of the Cartesian product reads as:

$$\gamma_k^d(G \square H) \le \gamma_k^d(G)n(H)\,.$$

By the well known commutativity property of the Cartesian product operation, this bound further implies that

$$\gamma_k^d(G \square H) \le \min\{\gamma_k^d(H)n(G), \gamma_k^d(G)n(H)\}.$$

The special case of the latter result for $k = 1$ has been recently obtained in [21].

3   Lexicographic products

The *lexicographic product* $G[H]$ of graphs G and H has $V(G[H]) = V_G \times V_H$, vertices (g_1, h_1) and (g_2, h_2) being adjacent if either $g_1 g_2 \in E_G$, or $g_1 = g_2$ and $h_1 h_2 \in E_H$.

Let S be a GDk-A in G with $k \ge 0$ and set $S = S_G \times V_H$. Since $|N_{S_G}(g)| - |N_{\bar{S}_G}(g)| \ge k$, we get that

$$|N_{S_G}(g)|n(H) - |N_{\bar{S}_G}(g)|n(H) \ge kn(H)\,,$$

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which in turn implies that for any vertex $h \in V_H$,

$$|N_{S_G}(g)|n(H) + \deg_H(h) - |N_{\bar{S}_G}(g)|n(H) \geq kn(H) + \delta_H \,.$$

Since $|N_{S_G}(g)|n(H) + \deg_H(h) = |N_S(g,h)|$ and $|N_{\bar{S}_G}(g)|n(H) = |N_{\bar{S}}(g,h)|$ this means that S is a global defensive $(kn(H) + \delta_H)$-alliance. As S is also a dominating set, we conclude that

$$\gamma^d_{kn(H)+\delta_H}(G[H]) \leq n(H)\gamma^d_k(G) \,.$$

For $k > 0$ this is a better result than

$$\gamma^d_k(G[H]) \leq n(H)\gamma^d_k(G)$$

because in such a case $k < kn(H) + \delta_H$, and so $\gamma^d_k(G[H]) \leq \gamma^d_{kn(H)+\delta_H}(G[H])$.

Eballe et al. [6] obtained some upper bounds of $\gamma^d_k(G[H])$ for the case $k = 0$ and $H = K_m$. In the next theorem, we present an upper bound on $\gamma^d_k(G[H])$ for the case when there exists a GDk-A in G with some special structure. For this sake recall that an *r-perfect code* in $G = (V_G, E_G)$ is a subset D of V_G for which the balls of radius r centered at the vertices of D form a partition of V_G, cf. [12].

Theorem 2. *Let $k > 0$, let S be a smallest GDk-A set in G, and suppose that $G\langle S\rangle$ has a 1-perfect code. If H is a graph with more than one vertex, then $G[H]$ has a GDk-A. Moreover, if $k \geq 2$, then*

$$\gamma^d_k(G[H]) \leq n(H)(\gamma^d_k(G) - \gamma(G\langle S\rangle)) + \gamma(G\langle S\rangle) \,.$$

Proof. First, suppose that $k > 2$. Let S_G be a GDk-A in G and D be a 1-perfect code in $G\langle S_G\rangle$. Then $|D| = \gamma(G\langle S_G\rangle)$. (This fact is well known and has been independently established several times, see [9, Theorem 9].)

Set $S = \big((S_G \setminus D) \times V_H\big) \cup \big(D \times \{v\}\big)$ where v is a vertex of minimum degree in H. It is easy to check that S is a dominating set in $G[H]$. We claim that S is a Dk-A in $G[H]$. To prove this claim, let $(g, h) \in S$. If $(g, h) \in \big((S_G \setminus D) \times V_H\big)$, then

$$
\begin{aligned}
|N_S((g,h))| - |N_{\bar{S}}((g,h))| &= (|N_S(g)| - 1)n(H) - (|N_{\bar{S}}(g)| + 1)n(H) + \deg_H(h) + 2 \\
&= n(H)(|N_S(g)| - |N_{\bar{S}}(g)|) - 2n(H) + \deg_H(h) + 2 \\
&\geq (k-2)n(H) + \deg_H(h) + 2 \geq k,
\end{aligned}
$$

which holds true because $k \geq 2$.

Consider next a vertex $(g, h) \in D \times \{v\}$. Then

$$
\begin{aligned}
|N_S((g,h))| - |N_{\bar{S}}((g,h))| &= |N_S(g)|n(H) - |N_{\bar{S}}(g)|n(H) - \delta_H \\
&= n(H)(|N_S(g)| - |N_{\bar{S}}(g)|) - \delta_H \\
&\geq kn(H) - \delta_H \geq k \,,
\end{aligned}
$$

which also holds for every $k \geq 2$.

We have thus proved that S is a Dk-A in $G[H]$. Moreover, since S contains the copy of S_G in the G-layer over v, by the definition of the lexicographic product we also infer that S is a dominating set. We conclude that S is a GDk-A in $G[H]$. Since clearly $|S| = n(H)(\gamma_k^d(G) - \gamma(G\langle S \rangle)) + \gamma(G\langle S \rangle)$, the proof is complete. $\square$

The proof of the bound from Theorem 2 does not work for $k = 1$. The reason is that the inequality $(k-2)n(H) + \deg_H(h) + 2 \geq k$ for the case $k = 1$ reduces to $-n(H) + \deg_H(h) + 2 \geq 1$ which clearly does not hold in general.

4 Corona and edge corona products

The *corona product* $G \circ H$ of graphs G and H is the graph obtained from the disjoint union of G and $n(G)$ copies of H bijectively assigned to the vertices of G, where each vertex $v \in V_G$ is adjacent to all the vertices of the assigned copy of H. This product was introduced in [7], see also [18, 19].

Theorem 3. *If G and H are graphs, then*

$$\gamma_k^d(G \circ H) \leq \min\{n(G)(1 + \gamma_{k-1}^d(H)), n(G)\gamma_{k+1}^d(H)\}.$$

Moreover, if $\delta_G - n(H) \geq k$, then $\gamma_k^d(G \circ H) = n(G)$.

Proof. Let G' be the subgraph of $G \circ H$ isomorphic to G and let H_i, $i \in [n(G')]$, be the the isomorphic copy of H corresponding to the ith vertex of G'.

Note that $\gamma(G \circ H) = n(G)$ and consequently $\gamma_k^d(G \circ H) \geq n(G)$. Hence if $\delta_G - n(H) \geq k$, then $V_{G'}$ is a Dk-A set in $G \circ H$ so that $\gamma_k^d(G \circ H) = n(G)$ holds in this case.

For the general case, consider an arbitrary defensive $(k-1)$-alliance S_H in H. Since each vertex of H_i has exactly one neighbor outside H_i, the set $(\cup_{i=1}^{n(G)} S_{H_i}) \cup V_{G'}$ is a GDk-A in $G \circ H$, where S_{H_i} is the copy of S_H in H_i. Therefore, $\gamma_k^d(G \circ H) \leq n(G) + n(G)\gamma_{k-1}(H) = n(G)(1 + \gamma_{k-1}(H))$.

Also, if S_H is a global defensive $(k+1)$-alliance in H, then from the same reasons as above, the set $\cup_{i=1}^{n(G)} S_{H_i}$ is a GDk-A in $G \circ H$. Hence $\gamma_k^d(G \circ H) \leq n(G)\gamma_{k+1}^d(H)$ and therefore, $\gamma_k^d(G \circ H) \leq \min\{n(G) + n(G)\gamma_{k-1}(H), n(G)\gamma_{k+1}^d(H)\}$. $\square$

Consider the corona products $G \circ K_m$, $m \geq 2$. Since $\gamma_1^d(K_m) = \lceil (m+2)/2 \rceil$ and $\gamma_{-1}^d(G) = \lfloor (m+1)/2 \rfloor$ (cf. [23]), Theorem 3, yields

$$\begin{aligned}
\gamma_0^d(G \circ K_m) &\leq& \min\{n(G)(1 + \gamma_{-1}^d(K_m)), n(G)\gamma_1^d(K_m)\} \\
&=& \min\{n(G)(1 + \lfloor (m+1)/2 \rfloor), n(G)\lceil (m+2)/2 \rceil\} \\
&=& n(G)\lceil (m+2)/2 \rceil.
\end{aligned}$$

On the other hand, it was proved in [6, Corollary 3.7] that if $m \geq 2$ and $\Delta(G) < m - 1$, then $\gamma_0^d(G \circ K_m) = n(G)\lceil(m+1)/2\rceil$. It follows that the bound of Theorem 3 is sharp for $G \circ K_m$ for all even m.

Another corona-like product was recently introduced as follows. The *edge corona* $G\diamondsuit H$ of graphs G and H is obtained by taking one copy of G and $m(G)$ disjoint copies of H associated to the edges of G, and for every edge $uv \in E_G$ joining u and v to every vertex of the copy of H associated to uv, see [11, 14]. For the statement of the next result recall that if S_G is a subset of vertices of a graph G, then its complement is denoted with $\bar{S}_G$.

Theorem 4. *Let G and H be two graphs. Then*

$$\gamma_k^d(G\diamondsuit H) \leq \min\{m(G)\gamma_{k+2}^d(H), \gamma_{k+n(H)\Delta_G}^d(G) + \gamma_{k+2}^d(H)|E_{G\langle\bar{S}_G\rangle}|,$$
$$\left(\gamma_{\lceil\frac{k}{n(H)+1}\rceil}^d(G) + n(H)|E_{G\langle S'_G\rangle}| + \gamma_{k+2}^d(H)|E_{G\langle\bar{S}'_G\rangle}|\right)I(k)\},$$

where S_G is a global defensive $(k+n(H)\Delta_G)$-alliance in G, S'_G is a global defensive $\lceil\frac{k}{n(H)+1}\rceil$-alliance in G, and

$$I(k) = \begin{cases} 1; & \text{if } k > -(n(H)+1), \\ \infty; & \text{otherwise}. \end{cases}$$

Proof. Let G' denote the copy of G in $G\diamondsuit H$, and let H_i be the copy of H corresponding to an edge $e_i \in E_G$.

If S_H is a global defensive $(k+2)$-alliance in H, then since each vertex of H_i has exactly two neighbors outside H_i, the set $\cup_{i=1}^{m(G)} S_{H_i}$, is a GD$k$-A in $G\diamondsuit H$ (again, S_{H_i} is the copy of S_H in H_i). Thus $\gamma_k^d(G\diamondsuit H) \leq n(G)\gamma_{k+2}^d(H)$.

Let next S_G be a global defensive $(k+n(H)\Delta_G)$-alliance in G, and S_H is a global defensive $(k+2)$-alliance in H. Since each vertex of H_i has exactly two neighbors outside H_i and as each vertex of G' has at most $\Delta_G n(H)$ neighbors outside G', the copy of S_G in G' together with the copies of S_H is each of the copies of H corresponding to the edges from $G\langle\bar{S}_G\rangle$ form a global defensive k-alliance in $G\diamondsuit H$. So, $\gamma_k^d(G\diamondsuit H) \leq \gamma_{k+n(H)\Delta_G}^d(G) + \gamma_{k+2}^d(H)|E_{G'\langle\bar{S}_G\rangle}|$.

Suppose now that S'_G is a global defensive $\lceil\frac{k}{n(H)+1}\rceil$-alliance in G. Let S_1 be the set of vertices of $G\diamondsuit H$ that lie in the copies of H corresponding to the edges of $G'\langle S'_{G'}\rangle$. In addition, in every copy of H corresponding to the edges from $\langle\bar{S}'_{G'}\rangle$ select a global defensive $(k+2)$-alliance in H. Then set $S = S'_{G'} \cup S_1 \cup S_2$. We claim that S is a GDk-A in $G\diamondsuit H$.

S is a dominating set in $G\diamondsuit H$, because $S'_{G'}$ dominates G' and all the copies of H above its edges as well as above copies of H above edges with one endpoint in $S'_{G'}$, while the other copies of H are dominated by S_2. To show that S is a Dk-A in

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$G \Diamond H$ consider $u \in S'_{G'}$. Then

$$
\begin{aligned}
|N_S(u)| - |N_{\bar{S}}(u)| &= (|N_{S'_G}(u)| - |N_{\bar{S}'_G}(u)|) \cdot (n(H) + 1) \\
&\geq \left\lceil \frac{k}{n(H) + 1} \right\rceil (n(H) + 1) \geq k.
\end{aligned}
$$

The same conclusion is clear when $u \in S \setminus S'_{G'}$. Therefore, S is a k-alliance in $G \Diamond H$ and so $\gamma_k^d(G \Diamond H) \leq \gamma_{\lceil \frac{k}{n(H)+1} \rceil}^d(G) + n(H)|E_{G\langle S'_G \rangle}| + \gamma_{k+2}^d(H)|E_{G\langle \bar{S}'_G \rangle}|$. □

The *sun graph* S_n is obtained by replacing every edge of a cycle C_n by a triangle C_3, cf. [5]. See Fig. 2 for S_3.
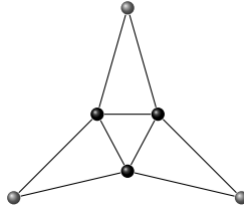

Figure 2: The sun graph S_3

From our point of view note that $S_n = C_n \Diamond K_1$. Since $\gamma_2^d(K_1) = \infty$, $\gamma_2^d(C_3) = 3$, $\gamma_0^d(C_3 = 2$, $|E_{C_3\langle S_G \rangle}| = 1$, $|E_{C_3\langle \bar{S}_G \rangle}| = 0$, and $I(0) = 1$, Theorem 4 implies that $\gamma_0^d(S_3) = \gamma_0^d(C_3 \Diamond P_1) \leq 3$. Actually, $\gamma_0^d(S_3) = 3$ (in Fig. 2 elements of a global defensive alliance in $C_3 \Diamond P_1$ are colored black), we have the inequality in Theorem 4.

References

[1] S.E. Anderson, Y. Guo, A. Tenney, K.A. Wash, Prime factorization and domination in the hierarchical product of graphs, Discuss. Math. Graph Theory 37 (2017) 873–890.

[2] S.E. Anderson, S. Nagpal, K. Wash, Domination in the hierarchical product and Vizing's conjecture, Discrete Math. 341 (2018) 20–24.

[3] M. Arezoomand, B. Taeri, Applications of generalized hierarchical product of graphs in computing the Szeged index of chemical graphs, MATCH Commun. Math. Comput. Chem. 64 (2010) 591–602.

[4] L. Barrière, C. Dalfó, M. A. Fiol, M. Mitjana, The generalized hierarchical product of graphs, Discrete Math. 309 (2009) 3871–3881.

[5] A. Brandstadt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM, Philadelphia, 1999.

[6] R.G. Eballe, R.M. Aldema, E.M. Paluga, R.F. Rulete, F.P. Jamil, Global defensive alliances in the join, corona and composition of graphs, Ars Combin. 107 (2012) 225–245.

[7] R. Frucht, F. Harary, On the coronas of two graphs, Aequationes Math. 4 (1970) 322–324.

[8] R. Hammack, W. Imrich, S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Boca Raton, FL, 2011.

[9] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[10] T.W. Haynes, D. Knisley, E. Seier, Y. Zou, A quantitative analysis of secondary RNA structure using domination based parameters on trees, BMC Bioinformatics 7 (2006) Pages/record No. 108.

[11] Y. Hou, W-C. Shiu, The spectrum of the edge corona of two graphs, Electron. J. Linear Algebra 20 (2010) 586–594.

[12] J. Kratochvíl, Perfect codes over graphs, J. Combin. Theory Ser. B 40 (1986) 224–228.

[13] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004) 157–177.

[14] Rinurwati, Slamin, H. Suprajitno, General results of local metric dimensions of edge-corona of graphs, Int. Math. Forum 11 (2016) 793–799.

[15] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global defensive k-alliances in graphs, Discrete Appl. Math. 157 (2009) 211–218.

[16] K.H. Shafique, R.D. Dutton, On satisfactory partitioning of graphs, Congr. Numer. 154 (2002) 183–194.

[17] M. Tavakoli, F. Rahbarnia, A.R. Ashrafi, Applications of the generalized hierarchical product of graphs in computing the vertex and edge PI indices of chemical graphs, Ric. Mat. 63 (2014) 59–65.

[18] M. Tavakoli, F. Rahbarnia, A.R. Ashrafi, Studying the corona product of graphs under some graph invariants, Trans. Combin. 3 (2014) 43–49.

[19] Y.-N. Yeh, I. Gutman, On the sum of all distances in composite graphs, Discrete Math. 135 (1994) 359–365.

[20] I.G. Yero, S. Bermudo, J.A. Rodríguez-Velázquez, J.M. Sigarreta, Partitioning a graph into defensive k-alliances, Acta Math. Sin. (Engl. Ser.) 27 (2011) 73–82.

[21] I.G. Yero, M. Jakovac, D. Kuziak, On global (strong) defensive alliances in some product graphs, Commun. Comb. Optim. 2 (2017) 21–33.

[22] I.G. Yero, J.A. Rodríguez-Velázquez, Computing global offensive alliances in Cartesian product graphs, Discrete Appl. Math. 161 (2013) 284–293.

[23] I.G. Yero, J.A. Rodríguez-Velázquez, A survey on alliances in graphs: Defensive alliances, Util. Math. 105 (2017) 141–172.